PULSE TRANSFER FUNCTION AND MANIPULATION OF BLOCK DIAGRAMS – Open loop systems

Digital control

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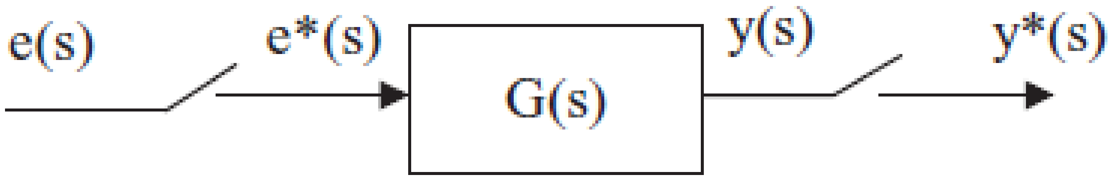
Pulse transfer function

- The pulse transfer function is the ratio of the z-transform of the sampled output and the input at the sampling instants.



$$y(s) = e^{*}(s)G(s).$$

$$y^{*}(s) = [e^{*}(s)G(s)]^{*} = e^{*}(s)G^{*}(s)$$

$$y(z) = e(z)G(z).$$

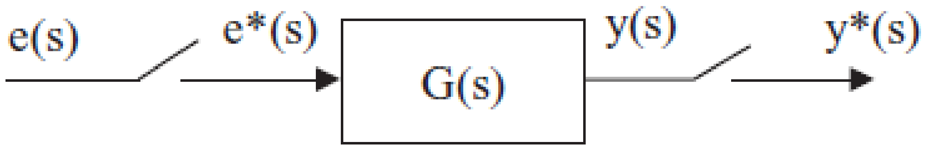
Pulse transfer function

$$y^*(s) = [e^*(s)G(s)]^* = e^*(s)G^*(s)$$

$$y(z) = e(z)G(z).$$

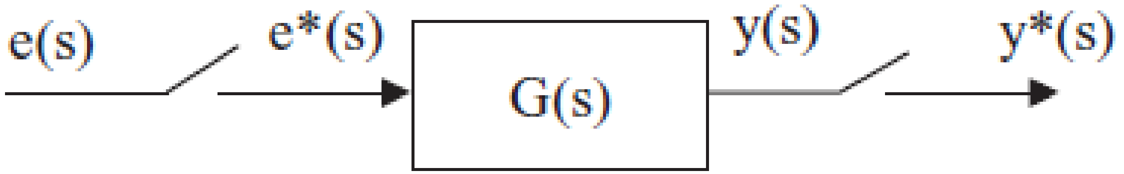
- If at least one of the continuous functions has been sampled, then the z-transform of the product is equal to the product of the z-transforms of each function (note that $[e^*(s)]^* = [e^*(s)]$, since sampling an already sampled signal has no further effect).
- $G(z)$ is the transfer function between the sampled input and the output at the sampling instants and is called the *pulse transfer function*.

Open-Loop Systems



- Example: Figure shows an open-loop sampled data system. Derive an expression for the z-transform of the output of the system.

Open-Loop Systems



- Solution

For this system we can write

$$y(s) = e^*(s)KG(s)$$

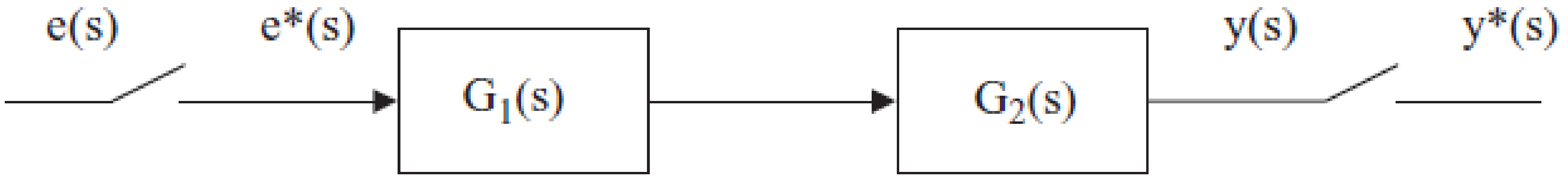
or

$$y^*(s) = [e^*(s)KG(s)]^* = e^*(s)KG^*(s)$$

and

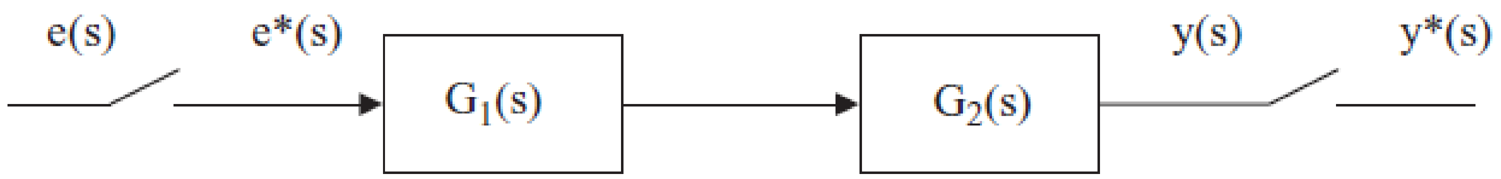
$$y(z) = e(z)KG(z).$$

Open-Loop Systems



- Derive an expression for the *z*-transform of the output of the system.

Open-Loop Systems



The following expressions can be written for the system:

$$y(s) = e^*(s)G_1(s)G_2(s)$$

or

$$y^*(s) = [e^*(s)G_1(s)G_2(s)]^* = e^*(s)[G_1G_2]^*(s)$$

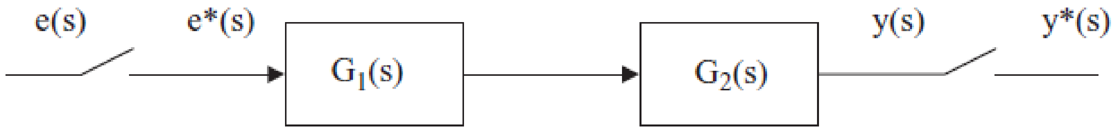
and

$$y(z) = e(z)G_1G_2(z),$$

where

$$G_1G_2(z) = Z\{G_1(s)G_2(s)\} \neq G_1(z)G_2(z).$$

Open-Loop Systems



For example, if

$$G_1(s) = \frac{1}{s}$$

and

$$G_2(s) = \frac{a}{s+a},$$

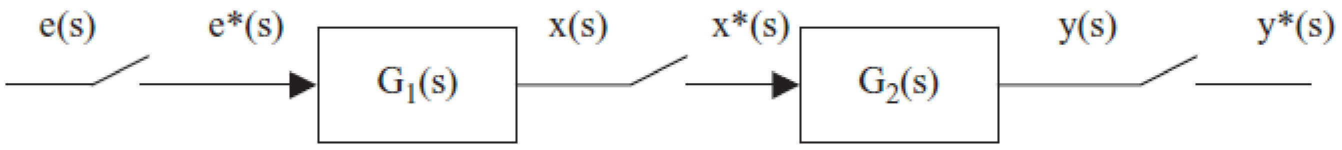
then from the z-transform tables,

$$Z\{G_1(s)G_2(s)\} = Z\left\{\frac{a}{s(s+a)}\right\} = \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$$

and the output is given by

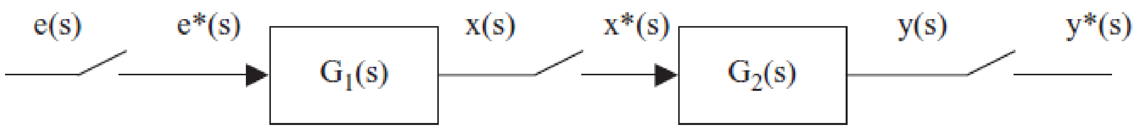
$$y(z) = e(z)\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}.$$

Open-Loop Systems



Derive an expression for the z-transform of the output of the system.

Open-Loop Systems



Solution

The following expressions can be written for the system:

$$x(s) = e^*(s)G_1(s)$$

or

$$x^*(s) = e^*(s)G_1^*(s), \tag{6.36}$$

and

$$y(s) = x^*(s)G_2(s)$$

or

$$y^*(s) = x^*(s)G_2^*(s). \tag{6.37}$$

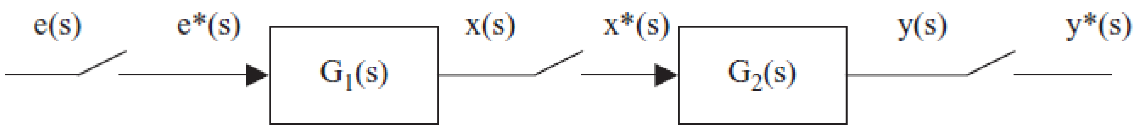
From (6.37) and (6.38),

$$y^*(s) = e^*(s)G_1^*(s)G_2^*(s),$$

which gives

$$y(z) = e(z)G_1(z)G_2(z).$$

Open-Loop Systems



For example, if

$$G_1(s) = \frac{1}{s} \quad \text{and} \quad G_2(s) = \frac{a}{s+a},$$

then

$$Z\{G_1(s)\} = \frac{z}{z-1} \quad \text{and} \quad Z\{G_2(s)\} = \frac{az}{z - ze^{-aT}},$$

and the output function is given by

$$y(z) = e(z)\frac{z}{z-1}\frac{az}{z - ze^{-aT}}$$

or

$$y(z) = e(z)\frac{az}{(z-1)(1-e^{-aT})}.$$

Open-Loop Time Response

- The open-loop time response of a sampled data system can be obtained by finding the inverse z-transform of the output function.

Example 6.18

A unit step signal is applied to the electrical RC system shown in Figure 6.21. Calculate and draw the output response of the system, assuming a sampling period of $T = 1$ s.

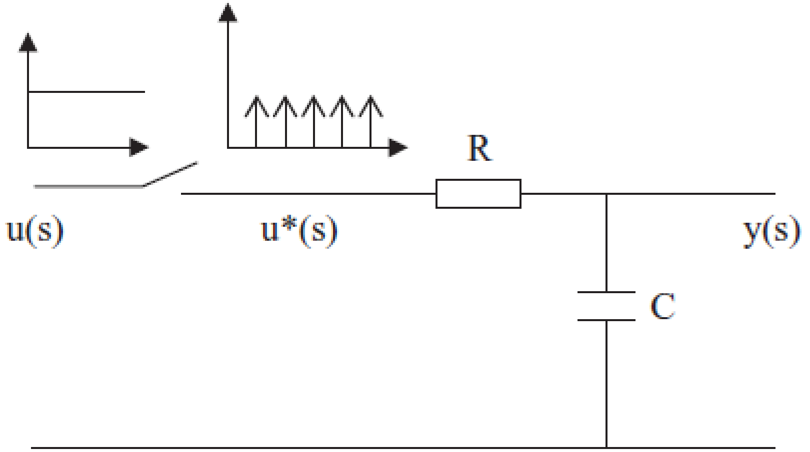


Figure 6.21 RC system with unit step input

Open-Loop Time Response

- The transfer function of the RC circuit can be found from KVL
- $0 = -RC\frac{dv}{dt} - Vc + Vs$
- Taking Laplace transform of the above equation leads to:
- $0 = -RCSVc(s) - Vc(S) + Vs(S)$
- $RC\ S\ Vc(s) + Vc(S) = Vs(S)$
- $\frac{Vc(s)}{Vs(s)} = \frac{1}{RCS+1}$

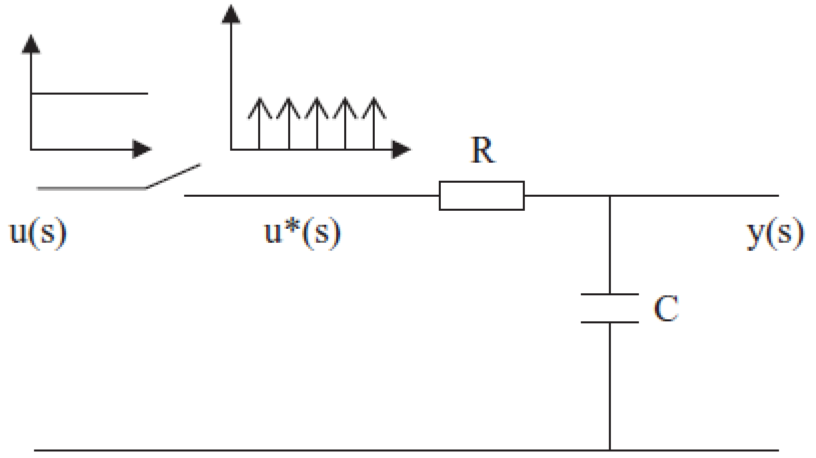


Figure 6.21 RC system with unit step input

Open-Loop Time Response

Solution

The transfer function of the RC system is

$$G(s) = \frac{1}{s+1}.$$

For this system we can write

$$y(s) = u^*(s)G(s)$$

and

$$y^*(s) = u^*(s)G^*(s),$$

and taking z-transforms gives

$$y(z) = u(z)G(z).$$

Open-Loop Time Response

The z-transform of a unit step function is

$$u(z) = \frac{z}{z-1}$$

and the z-transform of $G(s)$ is

$$G(z) = \frac{z}{z-e^{-T}}.$$

Thus, the output z-transform is given by

$$y(z) = u(z)G(z) = \frac{z}{z-1}\frac{z}{z-e^{-T}} = \frac{z^2}{(z-1)(z-e^{-T})};$$

since $T = 1$ s and $e^{-1} = 0.368$, we get

$$y(z) = \frac{z^2}{(z-1)(z-0.368)}.$$

Open-Loop Time Response

The output response can be obtained by finding the inverse z-transform of $y(z)$. Using partial fractions,

$$\frac{y(z)}{z} = \frac{A}{z-1} + \frac{B}{z-0.368}.$$

Calculating A and B, we find that

$$\frac{y(z)}{z} = \frac{1.582}{z-1} - \frac{0.582}{z-0.368}$$

or

$$y(z) = \frac{1.582z}{z-1} - \frac{0.582z}{z-0.368}.$$

Open-Loop Time Response

From the z-transform tables we find

$$y(nT) = 1.582 - 0.582(0.368)^n.$$

The first few output samples are

$$y(0) = 1,$$
$$y(1) = 1.367,$$
$$y(2) = 1.503,$$
$$y(3) = 1.552,$$
$$y(4) = 1.571,$$

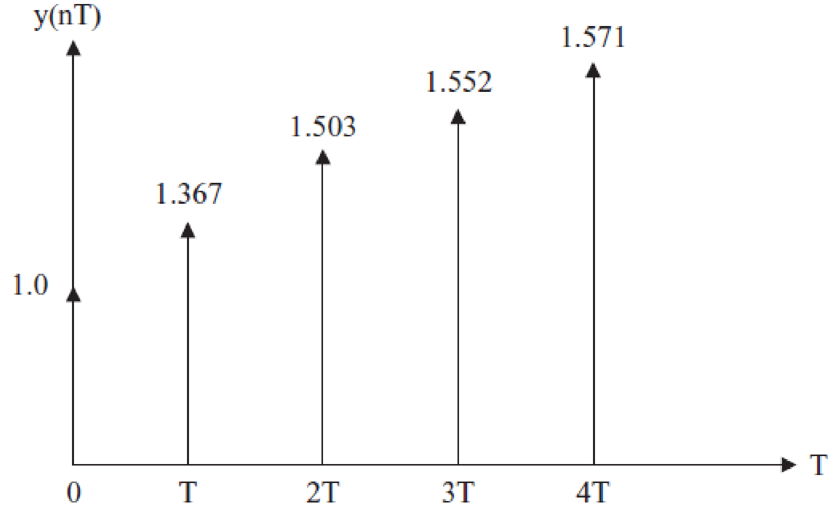


Figure 6.22 RC system output response

and the output response (shown in Figure 6.22) is given by

$$y(nT) = \delta(T) + 1.367\delta(t-T) + 1.503\delta(t-2T) + 1.552\delta(t-3T) + 1.571\delta(t-4T) + \ldots.$$

Open-Loop Time Response

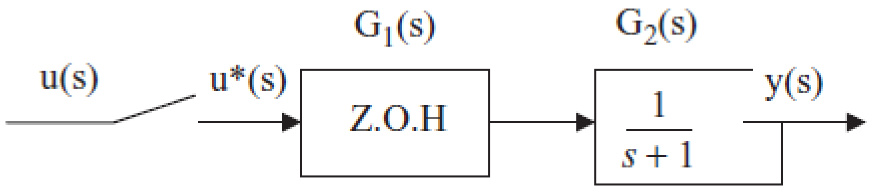


Figure 6.23 RC system with a zero-order hold

Example 6.19

Assume that the system in Example 6.17 is used with a zero-order hold (see Figure 6.23). What will the system output response be if (i) a unit step input is applied, and (ii) if a unit ramp input is applied.

Open-Loop Time Response

Solution

The transfer function of the zero-order hold is

$$G_1(s) = \frac{1 - e^{-Ts}}{s}$$

and that of the RC system is

$$G(s) = \frac{1}{s + 1}.$$

For this system we can write

$$y(s) = u^*(s) G_1 G_2(s)$$

and

$$y^*(s) = u^*(s)[G_1 G_2]^*(s)$$

or, taking z-transforms,

$$y(z) = u(z) G_1 G_2(z).$$

Open-Loop Time Response

Now, $T = 1$ s and

$$G_1G_2(s) = \frac{1 - e^{-s}}{s}\frac{1}{s+1},$$

and by partial fraction expansion we can write

$$G_1G_2(s) = (1 - e^{-s})\left(\frac{1}{s} - \frac{1}{s+1}\right).$$

From the z-transform tables we then find that

$$G_1G_2(z) = (1 - z^{-1})\left(\frac{z}{z-1} - \frac{z}{z-e^{-1}}\right) = \frac{0.63}{z - 0.37}.$$

Open-Loop Time Response

(i) For a unit step input,

$$u(z) = \frac{z}{z-1}$$

and the system output response is given by

$$y(z) = \frac{0.63z}{(z-1)(z-0.37)}.$$

Using the partial fractions method, we can write

$$\frac{y(z)}{z} = \frac{A}{z-1} + \frac{B}{z-0.37},$$

where $A = 1$ and $B = -1$; thus,

$$y(z) = \frac{z}{z-1} - \frac{z}{z-0.37}.$$

Open-Loop Time Response

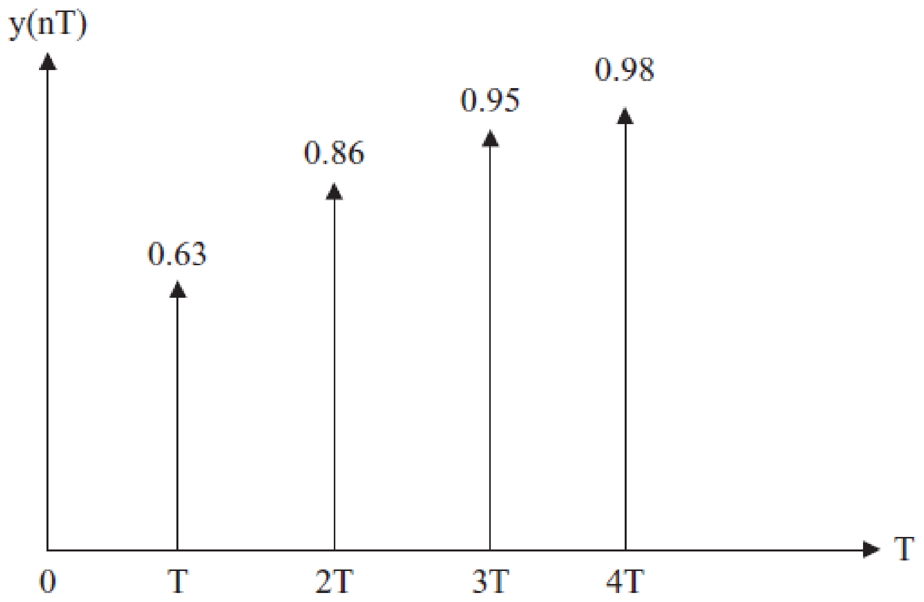


Figure 6.24 Step input time response of Example 6.19

From the inverse z-transform tables we find that the time response is given by

$$y(nT) = a - (0.37)^n,$$

where a is the unit step function; thus

$$y(nT) = 0.63\delta(t-1) + 0.86\delta(t-2) + 0.95\delta(t-3) + 0.98\delta(t-4) + \dots .$$

Open-Loop Time Response

(ii) For a unit ramp input,

$$u(z) = \frac{Tz}{(z-1)^2}$$

and the system output response (with $T = 1$) is given by

$$y(z) = \frac{0.63z}{(z-1)^2(z-0.37)} = \frac{0.63z}{z^3 - 2.37z^2 + 1.74z - 0.37}$$

Using the long division method, we obtain the first few output samples as

$$y(z) = 0.63z^{-2} + 1.5z^{-3} + 2.45z^{-4} + 3.43z^{-5} + \dots$$

and the output response is given as

$$y(nT) = 0.63\delta(t-2) + 1.5\delta(t-3) + 2.45\delta(t-4) + 3.43\delta(t-5) + \dots,$$

as shown in Figure 6.25.

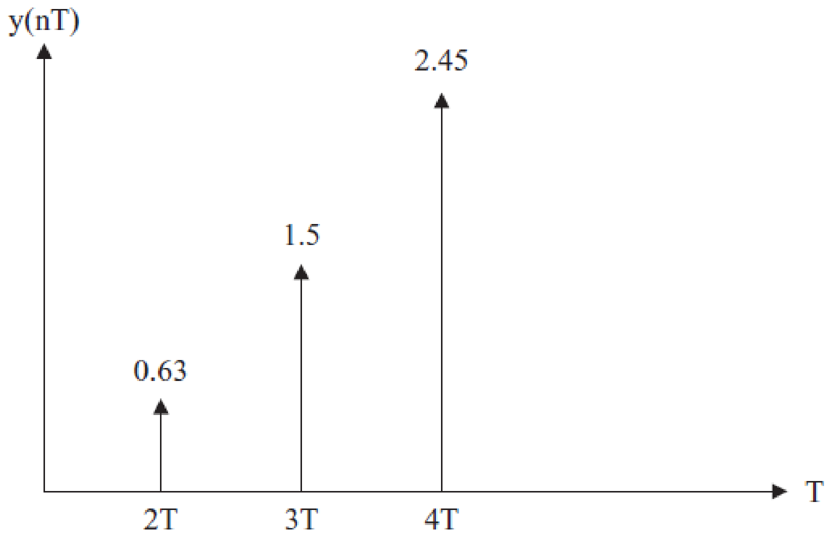


Figure 6.25 Ramp input time response of Example 6.19